Robinson Crusoe 1x1x1

Robinson Crusoe (RC) is the only living being on his island in the South Atlantic. He produces good C using his labor L as input. His utility function is

$$U = \ln C + \ln R$$

and his production function is

$$f(L) = A \cdot \sqrt{L}$$

His total time T is allocated between working (L) and resting (R), so that

$$T = L + R$$

1. Find the optimal values of L and C. Draw the situation
2. Find Robinson Crusoe's supply function as a producer who takes prices as given. Draw a graph
3. Solve Robinson Crusoe's problem as a consumer. Draw a graph
4. Find the equilibrium. Compare your result with the one obtained in part (1)

Solution

1. Robinson Crusoe chooses labor L and rest R in order to maximize

$$U = \ln C + \ln R$$

subject to the technological and time constraints

$$C = A\sqrt{L}$$

$$T = L + R$$

Substituting both constraints into the utility function, the problem becomes

$$\max_{L \in [0,T]} \ln\left(A\sqrt{L}\right) + \ln(T - L)$$

Equivalently,

$$\max_{L \in [0,T]} \ln A + \frac{1}{2}\ln L + \ln(T - L)$$

The first-order condition is

$$\frac{1}{2L} - \frac{1}{T - L} = 0$$

Thus,

$$T - L = 2L$$

$$T = 3L$$

$$L^* = \frac{T}{3}$$

Using the time constraint,

$$R^* = T - L^* = T - \frac{T}{3}$$

$$R^* = \frac{2T}{3}$$

Finally, optimal consumption is obtained from the production function

$$C^* = A\sqrt{L^*}$$

$$C^* = A\sqrt{\frac{T}{3}}$$

To verify that this is a maximum, note that

$$\frac{d^2U}{dL^2} = -\frac{1}{2L^2} - \frac{1}{(T - L)^2} < 0$$

so the objective function is strictly concave

Graphical interpretation

In the (R, C) space, the feasibility frontier is obtained from

$$L = T - R$$

so

$$C = A\sqrt{T - R}$$

This is the production possibility frontier between consumption and rest

Indifference curves are given by

$$\ln C + \ln R = \bar{U}$$

or equivalently

$$CR = \text{constant}$$

At the optimum, the highest attainable indifference curve is tangent to the frontier

$$(R^*, C^*) = \left(\frac{2T}{3}, A\sqrt{\frac{T}{3}}\right)$$

Therefore, Robinson Crusoe's optimal allocation is $L^* = \frac{T}{3}$, $R^* = \frac{2T}{3}$, and $C^* = A\sqrt{\frac{T}{3}}$

2. Now treat Robinson Crusoe as a firm that takes prices as given

   Let p be the price of consumption good C, and w the wage rate of labor L

   As a producer, Robinson Crusoe solves

$$\max_{L \geq 0} \pi(L) = pA\sqrt{L} - wL$$

   The first-order condition is

$$\frac{d\pi}{dL} = \frac{pA}{2\sqrt{L}} - w = 0$$

   Thus,

$$\frac{pA}{2\sqrt{L}} = w$$

$$\sqrt{L} = \frac{pA}{2w}$$

$$L^d(p, w) = \frac{A^2 p^2}{4w^2}$$

   This is Robinson Crusoe's labor demand as a producer

   Using the production function,

$$C^s = f(L^d) = A\sqrt{L^d}$$

we obtain the supply of good C

$$C^s(p,w) = A\sqrt{\frac{A^2p^2}{4w^2}}$$

$$C^s(p,w) = \frac{A^2p}{2w}$$

Since

$$\frac{d^2\pi}{dL^2} = -\frac{pA}{4L^{3/2}} < 0$$

the profit function is strictly concave, so this solution is indeed optimal

The corresponding profit level is

$$\pi^*(p,w) = pC^s - wL^d$$

$$\pi^*(p,w) = p\left(\frac{A^2p}{2w}\right) - w\left(\frac{A^2p^2}{4w^2}\right)$$

$$\pi^*(p,w) = \frac{A^2p^2}{4w}$$

Graphical interpretation

In the (L, C) plane, the production function is

$$C = A\sqrt{L}$$

which is increasing and concave

An isoprofit line is given by

$$\pi = pC - wL$$

or equivalently

$$C = \frac{\pi}{p} + \frac{w}{p}L$$

At the optimum, the highest attainable isoprofit line is tangent to the production function

The slope of the production function is

$$\frac{dC}{dL} = \frac{A}{2\sqrt{L}}$$

and the slope of the isoprofit line is

$$\frac{w}{p}$$

Tangency requires

$$\frac{A}{2\sqrt{L}} = \frac{w}{p}$$

which is exactly the first-order condition above

Thus, Robinson Crusoe's producer-side decisions are

$$L^d(p,w) = \frac{A^2p^2}{4w^2} \qquad C^s(p,w) = \frac{A^2p}{2w}$$

Therefore, Robinson Crusoe's supply function for good C is $C^s(p,w) = \frac{A^2p}{2w}$, and his labor demand is $L^d(p,w) = \frac{A^2p^2}{4w^2}$

3. Now treat Robinson Crusoe as a consumer who takes prices and income as given

He chooses consumption C and rest R to maximize

$$U = \ln C + \ln R$$

Let p be the price of good C, let w be the wage rate, and let π denote profit income from owning the firm

Since total time is T, the value of his full income is

$$wT + \pi$$

Hence, the consumer problem is

$$\max_{C,R} \ \ln C + \ln R$$

subject to

$$pC + wR \leq wT + \pi$$

The Lagrangian is

$$\mathcal{L} = \ln C + \ln R + \lambda\left(wT + \pi - pC - wR\right)$$

The first-order conditions are

$$\frac{\partial \mathcal{L}}{\partial C} = \frac{1}{C} - \lambda p = 0$$

$$\frac{\partial \mathcal{L}}{\partial R} = \frac{1}{R} - \lambda w = 0$$

$$\frac{\partial \mathcal{L}}{\partial \lambda} = wT + \pi - pC - wR = 0$$

From the first two conditions,

$$\lambda = \frac{1}{pC} = \frac{1}{wR}$$

Thus,

$$pC = wR$$

So Robinson Crusoe spends the same amount on consumption and on rest

Using the budget constraint,

$$pC + wR = wT + \pi$$

and since $pC = wR$, we get

$$2pC = wT + \pi$$

$$C^d(p, w, \pi) = \frac{wT + \pi}{2p}$$

Similarly,

$$2wR = wT + \pi$$

$$R^d(p, w, \pi) = \frac{wT + \pi}{2w}$$

If we want labor supply on the consumer side, we use

$$L^s = T - R^d$$

so

$$L^s = T - \frac{wT + \pi}{2w}$$

$$L^s(p, w, \pi) = \frac{wT - \pi}{2w}$$

Graphical interpretation

In the (R, C) plane, the budget constraint is

$$pC + wR = wT + \pi$$

or equivalently

$$C = \frac{wT + \pi}{p} - \frac{w}{p}R$$

This is a straight line with intercepts

$$\frac{wT + \pi}{p} \quad \text{and} \quad \frac{wT + \pi}{w}$$

and slope

$$-\frac{w}{p}$$

Indifference curves are given by

$$\ln C + \ln R = \bar{U}$$

or equivalently

$$CR = \text{constant}$$

At the optimum, the highest attainable indifference curve is tangent to the budget line

$$\frac{MU_R}{MU_C} = \frac{1/R}{1/C} = \frac{C}{R} = \frac{w}{p}$$

which is equivalent to

$$pC = wR$$

Thus, Robinson Crusoe's consumer-side demands are

$$C^d(p, w, \pi) = \frac{wT + \pi}{2p} \qquad R^d(p, w, \pi) = \frac{wT + \pi}{2w}$$

and his labor supply as a consumer is

$$L^s(p, w, \pi) = \frac{wT - \pi}{2w}$$

4. A competitive equilibrium in this Robinson Crusoe economy consists of prices (p, w), profit π, producer choices (L^d, C^s), and consumer choices (C^d, R^d) such that

$$L^d = T - R^d$$

$$C^s = C^d$$

and both the producer and the consumer solve their respective optimization problems

From part (2), Robinson Crusoe as a producer chooses

$$L^d(p, w) = \frac{A^2 p^2}{4w^2}$$

$$C^s(p, w) = \frac{A^2 p}{2w}$$

$$\pi(p, w) = \frac{A^2 p^2}{4w}$$

From part (3), Robinson Crusoe as a consumer chooses

$$C^d(p, w, \pi) = \frac{wT + \pi}{2p}$$

$$R^d(p, w, \pi) = \frac{wT + \pi}{2w}$$

so labor supply is

$$L^s(p, w, \pi) = T - R^d = \frac{wT - \pi}{2w}$$

In equilibrium, labor demand must equal labor supply

$$\frac{A^2p^2}{4w^2} = \frac{wT - \pi}{2w}$$

Substituting profits,

$$\pi = \frac{A^2p^2}{4w}$$

we get

$$\frac{A^2p^2}{4w^2} = \frac{wT - \frac{A^2p^2}{4w}}{2w}$$

$$\frac{A^2p^2}{4w^2} = \frac{T}{2} - \frac{A^2p^2}{8w^2}$$

Multiplying by $8w^2$,

$$2A^2p^2 = 4Tw^2 - A^2p^2$$

$$3A^2p^2 = 4Tw^2$$

Therefore, the equilibrium relative price satisfies

$$\frac{w}{p} = \frac{A\sqrt{3}}{2\sqrt{T}}$$

Since only relative prices matter, normalize

$$p = 1$$

Then

$$w^* = \frac{A\sqrt{3}}{2\sqrt{T}}$$

Now compute the equilibrium allocation

From producer labor demand,

$$L^* = \frac{A^2}{4(w^*)^2}$$

$$L^* = \frac{A^2}{4 \cdot \frac{3A^2}{4T}}$$

$$L^* = \frac{T}{3}$$

Hence,

$$R^* = T - L^*$$

$$R^* = \frac{2T}{3}$$

Output and consumption are

$$C^* = A\sqrt{L^*}$$

$$C^* = A\sqrt{\frac{T}{3}}$$

We can also verify goods-market clearing directly. From the consumer demand,

$$C^d = \frac{wT + \pi}{2}$$

and using

$$\pi = \frac{A^2}{4w^*}$$

we obtain

$$C^d = A\sqrt{\frac{T}{3}} = C^s$$

Thus, the competitive equilibrium allocation is

$$L^* = \frac{T}{3} \qquad R^* = \frac{2T}{3} \qquad C^* = A\sqrt{\frac{T}{3}}$$

with equilibrium relative price

$$\frac{w}{p} = \frac{A\sqrt{3}}{2\sqrt{T}}$$

This is exactly the same allocation obtained in part (1)

Therefore, the competitive equilibrium coincides with the direct Robinson Crusoe optimum from part (1)